

SEISMIC RESPONSE OF THE EARTH TO A GRAVITATIONAL WAVE IN THE 1-Hz BAND

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ABSTRACT

The response of an elastic solid to an incident gravitational wave is calculated in the linearized approximation to general relativity. The response depends on irregularities in the shear-wave modulus, and is strongest at free surfaces. Calculations are carried through in detail, first for a stationary flat Earth and then for a spherical rotating Earth. The seismic response on a rotating Earth is split into five frequencies, ω , $\omega \pm \Omega$, $\omega \pm 2\Omega$, where ω is the frequency of the incident wave and Ω is the Earth's rotation frequency. The seismic signals expected from plausible theoretical models of pulsars are about a factor of 10^6 below prevailing noise levels, but the uncertainties in this estimate are so great that an attempt to detect pulsars seismically is not hopeless.

I. INTRODUCTION

J. Weber has been the pioneer in examining both theoretically (Weber 1961, 1968*b*) and experimentally (Weber 1966, 1967, 1968*a*; Weber and Larson 1966) the detection on Earth of gravitational waves from astronomical sources. Until this year, attention has been concentrated on two widely separated wave bands, the kilohertz band and the millihertz band. The kilohertz band was studied because it can be monitored with self-contained detectors of reasonable size (Weber 1966) and because it is expected to be emitted copiously in some cataclysmic astronomical events. The millihertz band was studied because it can be monitored by observing the normal modes of vibration of the Earth, although no astronomical emission process of suitable intensity has been proposed.

The discovery of pulsating radio sources¹ led Weber to propose (Weber 1968*b*) a search for gravitational radiation on the 1-Hz band. No theoretical model of a pulsating radio source is generally accepted, but several possible models² involve rapidly moving objects of stellar mass and would be powerful gravitational radiators. The detection of gravitational waves from pulsating radio sources would be enormously helped if the periodicity of each source were precisely known. In principle, one needs only to accumulate data from existing seismometers over a long enough time, and to look for Fourier components at the pulsating radio source frequencies. Unfortunately, Weber (1968*b*) estimated that the amplitude of the seismic response would be too small to be detected against the background noise, even within the very narrow band width of the signal from the pulsating radio source. However, the uncertainties in any such estimate are very large; in the absence of a satisfactory theory of pulsating radio sources, a search for their periods in seismic data ought by all means to be encouraged.³

In the present paper we calculate in detail the response of the Earth to a gravitational wave in the 1-Hz band. We consider first a flat stationary Earth, and introduce later the effects of sphericity and rotation. The calculation goes beyond Weber (1968*b*) in allowing both compressive and shear-wave responses, and in including rotation. It turns out that the essential physics of the problem is already contained in the flat-Earth model. The

¹ For a recent review with references to the original papers, see Maran and Cameron (1968).

² For example, non-radial pulsations of a white-dwarf star, radial pulsations of a rotating star, or binary systems with short orbital period.

³ Dr Frank Press (private communication) has initiated such a search, so far with negative results.

flat-Earth model is good because the wavelength of seismic waves in the 1-Hz band is a few kilometers, small compared with the Earth's radius and large compared with the size of most surface irregularities. In the millihertz band the first of these two conditions would fail, and in the kilohertz band the second would fail.

The main qualitative result of the calculation is that, in an elastic medium, gravitational waves are absorbed only by irregularities of the shear-wave modulus. In a uniform medium there is no absorption. The Earth has two major discontinuities, the outer surface and the mantle-core interface. Other shear-wave irregularities are distributed in an unknown way through the interior. We carry through the calculation of seismic response, assuming a uniform homogeneous Earth. Thus we calculate the absorption of gravitational energy at the outer surface, neglecting all contributions from the interior. Since the detectors are at the surface, the surface absorption probably makes the dominant contribution to their motions. In any case, the neglect of inhomogeneities will make our estimates err on the low side.

It is possible to imagine arrangements of the local topography, in which a block of the Earth's crust may be seismically isolated by faults from its surroundings, and in which a gravitational wave may excite resonant standing vibrations. In such a case the response of a seismometer may exceed our estimate by a large factor, just as the amplitude of an ocean tide may be considerably amplified by local resonances in certain bays and channels.

In order to have a seismic response large enough to be observed, we must be lucky at both ends. We must have a source more powerful than we estimate, and we must also have a favorable geophysical configuration at the receiving end. The payoff from a successful observation would be correspondingly great. Any seismic detection of a pulsating radio source would immediately achieve three important objectives: (1) indisputable proof of the existence of gravitational waves; (2) independent evidence concerning the nature of pulsating radio sources; and (3) acquisition of a new tool for exploring the interior of the Earth.

II. FORMALISM

We consider an elastic solid whose equilibrium configuration is at rest in a Galilean coordinate system (x_0, x_1, x_2, x_3) with the Lorentz metric

$$-ds^2 = \eta^{\mu\nu} dx_\mu dx_\nu = -dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2. \quad (2.1)$$

Greek indices are summed from 0 to 3; Latin indices, from 1 to 3. Elastic motions of the solid are described by a 3-vector

$$z_j(x), \quad j = 1, 2, 3, \quad (2.2)$$

representing the displacement from equilibrium of the mass-point at position (x_1, x_2, x_3) and time $t = x_0/c$. The elastic properties of the medium are specified by a fourth-rank tensor

$$c_{jkmn} = c_{kijm} = c_{jknm} = c_{mnjk}, \quad (2.3)$$

with twenty-one independent components which may be functions of position.

We use non-relativistic mechanics to describe the motion of the solid, so that the Lagrangian is

$$\mathfrak{L} = T - V, \quad (2.4)$$

$$T = \frac{1}{2} \int \rho \dot{z}_j \dot{z}^j d\tau, \quad (2.5)$$

$$V = \frac{1}{2} \int c_{jkmn} z^{j,k} z^{m,n} d\tau, \quad (2.6)$$

where $\rho(x)$ is the density at the point x , and

$$z^{j,k} = (\partial z^j / \partial x_k) . \quad (2.7)$$

The energy-momentum tensor $T_{\mu\nu}$ has components

$$T_{00} = \frac{1}{2} \rho \dot{z}_j \dot{z}^j + \frac{1}{2} c_{jkmn} z^{j,k} z^{m,n} , \quad (2.8)$$

$$T_{0k} = \rho c \dot{z}_k , \quad (2.9)$$

$$T_{jk} = -S_{jk} = -c_{jkmn} z^{m,n} . \quad (2.10)$$

The law of conservation of momentum,

$$T_{j\mu}{}^{;\mu} = 0 , \quad (2.11)$$

gives the standard equation of motion of an elastic solid,

$$\frac{\partial}{\partial t} (\rho \dot{z}_j) = \frac{\partial}{\partial x_k} S_{jk} , \quad (2.12)$$

with the boundary condition

$$N^k S_{jk} = 0 , \quad (2.13)$$

where N^k is the vector normal at any point on the surface. It is necessary to spell out these elementary relations, in order to be sure that we have the correct energy-momentum tensor to express the interaction of the solid with a gravitational wave. The conservation law (2.11) expresses the invariance of the theory under space translations applied to the coordinates z^j which describe the positions of mass-elements. If we applied the usual field-theoretical recipe (Wentzel 1949) to the Lagrangian \mathfrak{L} , considering $z^j(x)$ as a classical field, we would obtain the wrong energy-momentum tensor. The usual procedure is based on translation invariance applied to the coordinates x_j , and is invalid in a situation where the coordinates x_j are anchored to a particular object.

A gravitational wave in the linear approximation of general relativity (Landau and Lifshitz 1962) is described by potentials

$$g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu} , \quad (2.14)$$

where $h^{\mu\nu}$ is extremely small compared with unity. If we work in the Hilbert gauge, defined by

$$h_{\mu}{}^{\nu}{}_{;\nu} = \frac{1}{2} h_{\nu}{}^{\nu}{}_{;\mu} , \quad (2.15)$$

then all components $h^{\mu\nu}$ satisfy the wave equation. A plane wave is represented by

$$h_{\mu\nu} = \text{Re} [a e_{\mu\nu} \exp (ik^{\mu} x_{\mu})] , \quad (2.16)$$

with a frequency ω given by

$$\omega = ck_0 = -ck^0 = c(k_j k^j)^{1/2} . \quad (2.17)$$

The complex amplitude a defines the intensity and phase of the wave, and the polarization tensor $e_{\mu\nu}$ can be chosen in the normalized form

$$e_{0\mu} = e_{\mu 0} = 0 , \quad (2.18)$$

$$e_{jk} = u_l L_j L_k + u_r R_j R_k , \quad (2.19)$$

$$|u_l|^2 + |u_r|^2 = 1 , \quad (2.20)$$

where u_l, u_r are the amplitudes of left-handed and right-handed circularly polarized components. The standard polarization vectors L_j and R_j are complex unit vectors satisfying the conditions

$$R_j = L_j^*, \quad R_j L_j = 1, \quad (2.21)$$

$$R_j R_j = L_j L_j = R_j k_j = L_j k_j = 0. \quad (2.22)$$

In linearized gravitation theory, the interaction between the wave (2.14) and the elastic solid adds a term

$$\mathfrak{L}' = -\frac{1}{2} h^{\mu\nu} T_{\mu\nu} \quad (2.23)$$

to the Lagrangian \mathfrak{L} . It is easy to verify that, even if $e_{\mu\nu}$ is not chosen in the special form (2.18), the zero-components of $h^{\mu\nu}$ contribute nothing to \mathfrak{L}' by virtue of the gauge-conditions (2.15) and the conservation-law (2.11). Thus \mathfrak{L}' reduces to

$$\mathfrak{L}' = \frac{1}{2} h^{jk} S_{jk}. \quad (2.24)$$

The elastic equations (2.12) and (2.13) are thereby modified to

$$\frac{\partial}{\partial t} (\rho \dot{z}_j) = \frac{\partial}{\partial x_k} [c_{jkmn} (z^{m,n} - \frac{1}{2} h^{mn})], \quad (2.25)$$

with the boundary condition

$$N^k [c_{jkmn} (z^{m,n} - \frac{1}{2} h^{mn})] = 0 \quad (2.26)$$

on the surface.

An important special case of these equations occurs when the solid is isotropic. Then (Love 1927)

$$c_{jkmn} = \lambda \delta_{jk} \delta_{mn} + \mu (\delta_{jm} \delta_{kn} + \delta_{jn} \delta_{km}), \quad (2.27)$$

with elastic moduli λ, μ which may still be functions of position. In fact,

$$\mu = \rho s^2, \quad \lambda = \rho (v^2 - 2s^2), \quad (2.28)$$

where s, v are the local velocities of shear waves and compressive waves, respectively. Since the tensor h^{mn} is traceless and divergenceless,

$$\delta_{mn} h^{mn} = (\partial/\partial x_k) h^{mk} = 0, \quad (2.29)$$

the equation of motion (2.25) reduces to

$$\frac{\partial}{\partial t} (\rho \dot{z}_j) = \frac{\partial}{\partial x_j} (\lambda z^{m,m}) + \frac{\partial}{\partial x_k} [\mu (z^{j,k} + z^{k,j})] - \left(\frac{\partial \mu}{\partial x_k} \right) h^{jk}. \quad (2.30)$$

This shows directly that a gravitational wave in the interior of an isotropic elastic medium interacts only with irregularities in the shear modulus μ . The boundary condition (2.26) becomes

$$\lambda N_j z^{m,m} + \mu N_k (z^{j,k} + z^{k,j} - h^{jk}) = 0, \quad (2.31)$$

showing that the surface interaction also depends only on the local value of μ at the surface.

Going back to the general equations (2.25) and (2.26), we see that the gravitational forces are always independent of the z^j and are simply added as an inhomogeneous driving term to the linear elastic equations. The response of the Earth to a gravitational wave can in principle be calculated exactly, if we know the normal modes of vibration of

the Earth, by determining the coupling of the driving potential h^{mn} into each normal mode. In the millihertz band this would be a practical method of calculation; in the 1-Hz band the number of normal modes is so large that it is easier to solve equation (2.25) directly.

Another useful feature of equations (2.25) and (2.26) is that the acceleration of any mass-element is a linear function of the local elastic coefficients c_{jkmn} . In particular, any free mass will have zero acceleration. The coordinate system (x_0, x_1, x_2, x_3) remains an inertial system for slowly moving masses, even in the presence of gravitational waves.

A seismometer consists ideally of a mass M loosely connected to its surroundings by weak springs. In the equation of motion (2.25) for the mass M , the term in h^{mn} multiplied by a weak-spring constant is negligible. The mass M is thus unaffected by the gravitational wave, while the surroundings move according to the local amplitude of the Earth's response. The seismometer records the relative displacement of the mass M from its surroundings. This displacement, after making allowance for the spectral response of the instrument, gives a direct measurement of the amplitude z^j of the Earth's motion in the vicinity.

III. FLAT-EARTH MODEL

As a rough approximation to describe the configuration of the Earth in the neighborhood of a given point on the surface, consider an infinite half-space $x_3 > 0$ filled with a uniform isotropic elastic medium. Insofar as the response of the Earth to a gravitational wave is mainly a localized surface effect, the flat-Earth model will be qualitatively correct. The uniformity and isotropy of the medium mean that the elastic velocities s and v are constants, so that the coupling term in equation (2.30) vanishes. There is no effect at all of a gravitational wave in the interior of the medium. The solutions of equation (2.30) are plane waves running away from the surface with velocities s and v . Incoming elastic waves are excluded, since we are neglecting signals propagating through the Earth from the other side. The boundary condition (2.31) on the surface $x_3 = 0$ becomes

$$\lambda \delta_{j3} z^{m,m} + \mu (z^{j,3} + z^{3,j}) = \mu h^{3j}. \quad (3.1)$$

Suppose that we have a gravitational wave with frequency ω and wave-vector

$$k^j = (\omega/c)[\sin \theta, 0, \cos \theta], \quad (3.2)$$

incident at an angle θ to the inward normal. Let the wave be pure right-handed circularly polarized. The gravitational potentials are then, according to equations (2.16)–(2.22),

$$h_{jk} = \text{Re} [a R_j R_k \exp (i k^j x_j - i \omega t)], \quad (3.3)$$

with

$$R_j = 2^{-1/2} [\cos \theta, i, -\sin \theta]. \quad (3.4)$$

The elastic response of the medium will be a superposition of two plane waves, one longitudinal and one transverse,

$$z^j = \text{Re} [y_L^j \exp (i p^j x_j - i \omega t) + y_T^j \exp (i q^j x_j - i \omega t)], \quad (3.5)$$

with

$$p^j p_j = (\omega^2/v^2), \quad q^j q_j = (\omega^2/s^2), \quad (3.6)$$

$$y_L^j = a p^j, \quad y_T^j q_j = 0. \quad (3.7)$$

The boundary condition becomes

$$p_1 = q_1 = k_1, \quad p_2 = q_2 = k_2, \quad (3.8)$$

$$i \lambda \delta_{j3} (y_L^m p_m) + i \mu (y_L^j p_3 + y_L^3 p_j + y_T^j q_3 + y_T^3 q_j) = a \mu R_3 R_j. \quad (3.9)$$

Now we use the fact that s and v are enormously less than the velocity of light c . By equations (3.8), the components p_1, p_2, q_1, q_2 are all of the order of magnitude (ω/c) , while p_3, q_3 are by equations (3.6) of the order (ω/v) and (ω/s) . Thus the elastic waves propagate perpendicularly to the surface to a very good approximation. The relation between the incident gravitational wave and the emergent elastic wave is identical with the relation between incident and refracted waves in an optical medium with enormously high refractive index.

When we neglect p_1, p_2, q_1, q_2 compared with p_3, q_3 , equation (3.5) gives

$$p_3 = (\omega/v), \quad q_3 = (\omega/s), \quad (3.10)$$

and equation (3.7) becomes

$$y_L^1 = y_L^2 = 0, \quad y_T^3 = 0. \quad (3.11)$$

We write then simply y^j for the vector whose (1,2)-components are y_T and whose 3-component is y_L . The boundary condition (3.9) splits into separate equations for the three components, namely,

$$\begin{aligned} y^j &= -ia(s/\omega)[R_3 R_1, R_3 R_2, (s/v)R_3 R_3] \\ &= \frac{1}{2}ia(s/\omega) \sin \theta [\cos \theta, i, - (s/v) \sin \theta]. \end{aligned} \quad (3.12)$$

Equation (3.12) gives complete information concerning the phase, amplitude, and direction of the seismic displacement induced at the Earth's surface by a gravitational wave. In this approximation all points on the surface move together in phase. Thus all the instruments in a seismic array should respond coherently to a gravitational wave, provided that the extent of the array is small compared with the free-space wavelength $(2\pi c/\omega)$, a condition which is well satisfied in practice for waves in the 1-Hz band.

The mass-elements on the Earth's surface according to equation (3.12) describe elliptical orbits with the major axes in the 2-direction. If the incident wave is left-circularly polarized, only the sign of the 2-component in equation (3.12) is reversed. The response to a linearly polarized wave is obtained in an obvious way by taking a linear superposition of the two circularly polarized responses.

The flux of energy carried into the Earth by the elastic wave (3.5) is

$$Q = \frac{1}{2}\rho\omega^2[s(|y^1|^2 + |y^2|^2) + v|y^3|^2] \quad (3.13)$$

per unit area of surface. For the case of a circularly polarized incident wave, equation (3.12) gives

$$Q = \frac{1}{8}\rho s^3 |a|^2 \sin^2 \theta [1 + \cos^2 \theta + (s/v) \sin^2 \theta]. \quad (3.14)$$

This has to be compared with the energy flux in the incident gravitational wave (2.16), which is (Landau and Lifshitz 1962)

$$F = \frac{c^3 \omega^2 |a|^2}{64\pi G} |\cos \theta| \quad (3.15)$$

per unit surface area. Hence the proportion of the energy crossing the surface which is converted into elastic energy is

$$\epsilon = \frac{Q}{F} = \left(\frac{8\pi G \rho}{\omega^2}\right) \left(\frac{s}{c}\right)^3 \frac{\sin^2 \theta}{|\cos \theta|} [1 + \cos^2 \theta + (s/v) \sin^2 \theta]. \quad (3.16)$$

The same formula holds, whether the gravitational wave is incident on the surface from outside or from inside. For a linearly polarized wave, in which the polarization

tensor e_{jk} has a principal axis making an angle ϕ with the 2-direction, the factor in brackets in equation (3.16) should be replaced by

$$2 \sin^2 2\phi + 2[\cos^2 \theta + (s/v) \sin^2 \theta] \cos^2 2\phi. \quad (3.17)$$

For the Earth, $8\pi G\rho$ is about $5 \times 10^{-6} \text{ sec}^{-2}$, while s is about $4.5 \times 10^5 \text{ cm sec}^{-1}$. For waves of 1-sec period, $\omega = 2\pi \text{ sec}^{-1}$, so that the first two factors in equation (3.16) are

$$\frac{8\pi G\rho}{\omega^2} \approx 10^{-7}, \quad \left(\frac{s}{c}\right)^3 \approx 4 \times 10^{-15}, \quad (3.18)$$

and

$$\epsilon \approx 10^{-21}. \quad (3.19)$$

This very small efficiency of energy conversion is mainly due to the extreme mismatch between the velocities s and c .

The quantity directly relevant to observation is the displacement y^j given by equation (3.12). The numerical value of y^j depends on the amplitude a of the incident wave. A reasonable order of magnitude for a is obtained by assuming the source of gravitational waves to be equal in energy flux to a star of bolometric magnitude zero. Plausible dynamical models⁴ can put out this amount of energy without difficulty. The energy flux of a magnitude-zero source is (Allen 1955)

$$\frac{c^3 \omega^2 |a|^2}{64\pi G} = 2 \times 10^{-5} \text{ erg cm}^{-2} \text{ sec}^{-1}, \quad (3.20)$$

which implies

$$|a|\omega = 3 \times 10^{-21} \text{ sec}^{-1}. \quad (3.21)$$

The displacement in the 2-direction (horizontal) produced by a horizontally incident wave ($\theta = \frac{1}{2}\pi$) according to equation (3.12) is then

$$y = (as/2\omega) = 1.5 \times 10^{-21}(s/\omega^2). \quad (3.22)$$

If we take $s = 4.5 \times 10^5 \text{ cm sec}^{-1}$, $\omega = 6 \text{ sec}^{-1}$, we find

$$y = 2 \times 10^{-17} \text{ cm}. \quad (3.23)$$

This result, considering the rough nature of our assumptions, is in satisfactory agreement with Weber's (1968*b*) estimate.

It is noteworthy that the dissipation of seismic energy in the Earth plays no role in our analysis. There are, of course, dissipative processes which should be included in the equation of motion (2.25) in an exact theory. But in the flat-Earth model it makes no difference to the elastic displacements at the surface whether the ingoing elastic waves propagate freely to infinity or are dissipated in the interior. The details of the dissipative processes would be important only if the surface effects could be amplified by reflected waves from the interior. If reflections in the Earth are important, standing-wave patterns can exist, and the amplitude of seismic response will depend on details of the local topography and of local damping effects.

IV. EFFECTS OF SPHERICITY AND ROTATION

The normal modes of vibration of a uniform isotropic solid sphere have been fully described by Love (1927). However, it is pointless to use these modes in the present context, since the Earth as a whole is not even approximately uniform. It is reasonable to consider the Earth as approximately uniform within a distance of a few tens of kilometers from a given point on the surface, but within this distance it is also a good ap-

⁴ See n. 2.

proximation to consider the Earth flat. A uniform-sphere model would have no advantage over the much simpler flat-Earth model.

In this section we shall calculate the seismic response to a gravitational wave on a spherical rotating Earth, assuming that the local dynamics is described by the flat-Earth equations. We consider only the kinematical effects of sphericity and rotation, produced by the motion of the source relative to an Earth-bound detector. The effects of rotation would be qualitatively similar if we replaced the flat-Earth model by a more exact dynamical theory. The main features of the rotation effects arise from purely group-theoretical considerations.

Let (ξ_1, ξ_2, ξ_3) be a coordinate system with origin at the center of the Earth, the 3-direction pointing to the North Pole and the (1,2)-directions fixed in space. Let (x_1, x_2, x_3) be a coordinate system with origin at a point O on the Earth's surface and rotating with the Earth, x_3 being vertically downward, x_1 horizontal north, x_2 horizontal east. Let R, Ω be the Earth's radius and angular velocity, and let α be the latitude of O . The relation between the two coordinate systems is

$$\xi_j = M_{jk}x_k - M_{j3}R, \quad (4.1)$$

$$x_j - R\delta_{j3} = M_{kj}\xi_k, \quad (4.2)$$

where M_{jk} is the matrix

$$M = \begin{pmatrix} -\sin \alpha \cos \Omega t & -\sin \Omega t & -\cos \alpha \cos \Omega t \\ -\sin \alpha \sin \Omega t & \cos \Omega t & -\cos \alpha \sin \Omega t \\ \cos \alpha & 0 & -\sin \alpha \end{pmatrix}. \quad (4.3)$$

Suppose that there is a source of pure right-circularly polarized gravitational waves at declination β . The gravitational potentials in the ξ -system are then

$$h_{jk}(\xi) = \text{Re} \{aR_jR_k \exp [(-i\omega/c)(\xi_1 \cos \beta + \xi_3 \sin \beta + ct)]\}, \quad (4.4)$$

$$R_j = 2^{-1/2}(-\sin \beta, i, \cos \beta). \quad (4.5)$$

The potentials must be expressed in terms of the x -coordinates before the theory of the previous section can be applied. Thus the potentials observed with Earth-bound equipment at the point O are

$$h_{jk}(O) = \text{Re} \{aM_{mj}M_{nk}R_mR_n \times \exp [(-i\omega/c)(R \cos \alpha \cos \beta \cos \Omega t + R \sin \alpha \sin \beta + ct)]\}. \quad (4.6)$$

There are two kinematical effects of rotation represented in the potentials (4.6). The first is the Doppler effect produced by the motion of the detector at O relative to the distant source. The Doppler effect contributes the factor

$$\exp(-i\gamma \cos \Omega t), \quad \gamma = (\omega R/c) \cos \alpha \cos \beta, \quad (4.7)$$

in equation (4.6). Now

$$\exp(-i\gamma \cos \Omega t) = \sum_{n=-\infty}^{\infty} (-i)^n J_n(\gamma) \exp(in\Omega t). \quad (4.8)$$

The Doppler effect splits the incident wave with frequency ω into a superposition of waves with frequencies $\omega, \omega \pm \Omega, \omega \pm 2\Omega, \dots$, the component $(\omega + n\Omega)$ having amplitude proportional to $J_n(\gamma)$. The splitting is unimportant for waves in the 1-Hz band, since then

$$\gamma \leq (\omega R/c) \approx 0.1, \quad (4.9)$$

while

$$|J_n(\gamma)| < [(\frac{1}{2}\gamma)^n/n!] . \quad (4.10)$$

If we are considering waves with periods shorter than a tenth of a second, the rotational Doppler splitting would be substantial.

The second kinematical effect of rotation is produced by the apparent motion of the source in the sky, and is expressed in equation (4.6) by the time-dependent coefficients M_{mj} . The apparent-motion effect is independent of the Doppler effect, and we may isolate the apparent-motion effect by dropping the factor (4.7) from the potentials (4.6). We assume, then, an incident wave with potentials

$$h_{jk} = \text{Re} [a M_{mj} M_{nk} R_m R_n \exp(-i\omega t)] , \quad (4.11)$$

as observed by detectors rotating with the Earth at the point O . According to the theory of § 3, the displacement at O relative to the Earth-fixed coordinate system is

$$z^j = \text{Re} [y^j \exp(-i\omega t)] , \quad (4.12)$$

$$y^j = -ia(s/\omega) M_{m3} R_m R_n [M_{n1}, M_{n2}, (s/v) M_{n3}] . \quad (4.13)$$

We are here assuming that the flat-Earth model gives a valid description of the dynamics of earth motions in the region adjacent to the point O .

To complete the analysis of equation (4.13), it is convenient to resolve the polarization tensor $R_m R_n$ into five parts, each part labeled by an index $p = 2, 1, 0, -1, -2$, which represents the component of "angular momentum" about the Earth's rotation axis. From equation (4.5) we find

$$R_m R_n = \sum_{p=-2}^2 \binom{4}{2+p}^{1/2} (\cos^{2-p} \delta) (\sin^{2+p} \delta) G_{mn}^p , \quad (4.14)$$

$$\delta = \frac{1}{2} \left(\frac{\pi}{2} - \beta \right) , \quad (4.15)$$

with the five standard polarization tensors now referred to the 3-axis,

$$G^{\pm 2} = \frac{1}{2} \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad G^{\pm 1} = \frac{1}{2} \begin{pmatrix} 0 & 0 & \pm 1 \\ 0 & 0 & i \\ \pm 1 & i & 0 \end{pmatrix} , \quad (4.16)$$

$$G^0 = 6^{-1/2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

From equations (4.3) and (4.14) we find

$$M_{m3} M_{nj} R_m R_n = \sum_{p=-2}^2 (\cos^{2-p} \delta) (\sin^{2+p} \delta) \exp(ip\Omega t) V_j^p , \quad (4.17)$$

with vectors V^p depending on α alone. Hence the displacement (4.13) becomes

$$y^j = -ia(s/\omega) \sum_{p=-2}^2 (\cos^{2-p} \delta) (\sin^{2+p} \delta) \exp(ip\Omega t) U_j^p , \quad (4.18)$$

with the vectors U^p given by

$$U^{\pm 2} = \frac{1}{2}[\cos \alpha \sin \alpha, \mp i \cos \alpha, (s/v) \cos^2 \alpha], \quad (4.19)$$

$$U^{\pm 1} = [\mp \cos 2\alpha, -i \sin \alpha, \pm (s/v) \sin 2\alpha], \quad (4.20)$$

$$U^0 = [-3 \sin \alpha \cos \alpha, 0, (s/v)(3 \sin^2 \alpha - 1)]. \quad (4.21)$$

The incident wave with frequency ω produces a seismic response split into five components with frequencies ω , $\omega \pm \Omega$, $\omega \pm 2\Omega$. Equations (4.19)–(4.21) give precisely the relative amplitude, phase, and orientation of the five components. These formulae apply to a pure right-circularly polarized incident wave; for left-circular polarization, the same formulae hold with β replaced by $(-\beta)$ and the signs of $U^{\pm 1}$ reversed. In cases where the Doppler splitting is significant, each of the five components of the displacement appearing in equation (4.18) must be further split by Doppler effect according to equation (4.8).

Considering only the apparent-motion splitting, we calculate the energy flux going into each of the five components of equation (4.18). According to equation (3.13), the flux in the seismic wave of frequency $(\omega - p\Omega)$ per unit area of Earth at latitude α is

$$Q_p = (1/32)\rho s^3 |a|^2 (1 + \sin \beta)^{2-p} (1 - \sin \beta)^{2+p} \Gamma_p, \quad (4.22)$$

with

$$\Gamma_{\pm 2} = \frac{1}{4} \cos^2 \alpha (1 + \sin^2 \alpha + (s/v) \cos^2 \alpha), \quad (4.23)$$

$$\Gamma_{\pm 1} = \cos^2 2\alpha + \sin^2 \alpha + (s/v) \sin^2 2\alpha, \quad (4.24)$$

$$\Gamma_0 = 9 \sin^2 \alpha \cos^2 \alpha + (s/v)(3 \sin^2 \alpha - 1)^2. \quad (4.25)$$

If we integrate this energy flux over the entire surface of the Earth, we find the total energy absorbed per second into the frequency $(\omega - p\Omega)$ to be

$$E_p = (\pi R^2/40)\rho s^3 [1 + \frac{2}{3}(s/v)] |a|^2 \binom{4}{2+p} (1 + \sin \beta)^{2-p} (1 - \sin \beta)^{2+p}. \quad (4.26)$$

Considering that the incident flux is given by equation (3.15), the rotating Earth presents a cross-section

$$\sigma_p = \left(\frac{\pi R^2}{5}\right) \left(\frac{8\pi G\rho}{\omega^2}\right) \left(\frac{s}{c}\right)^3 \left(1 + \frac{2s}{3v}\right) \binom{4}{2+p} (1 + \sin \beta)^{2-p} (1 - \sin \beta)^{2+p}, \quad (4.27)$$

for absorption of a right-circularly polarized gravitational wave from a source at declination β into a seismic wave at frequency $(\omega - p\Omega)$. Summing equation (4.27) over the five seismic frequencies, we find a total cross-section

$$\sigma_T = \left(\frac{16\pi R^2}{5}\right) \left(\frac{8\pi G\rho}{\omega^2}\right) \left(\frac{s}{c}\right)^3 \left(1 + \frac{2s}{3v}\right), \quad (4.28)$$

independent of the declination of the source. The same cross-section is obtained for a non-rotating Earth by integrating equation (3.16). If, instead of summing over the five frequencies, we choose a particular component p and average the cross-section (4.27) over all positions of the source in the sky, we find

$$\langle \sigma_p \rangle = \frac{1}{5} \sigma_T, \quad (4.29)$$

independent of p . A random distribution of sources puts equal amounts of energy into each of the five frequencies ω , $\omega \pm \Omega$, $\omega \pm 2\Omega$. An order-of-magnitude estimate of the

cross-section (4.28), using the numerical ratios (3.18), is

$$\sigma_T \approx 10^{-3} \text{ cm}^2 \quad (4.30)$$

for radiation in the 1-Hz band.

The theory of this and the preceding sections applies almost without change to the absorption of a gravitational wave at the core-mantle interface. At a liquid-solid interface all the equations remain valid, except that the fraction (s/v) appearing in equations (3.12), (3.14), (3.16), (3.17), (4.13), and (4.19)–(4.28) should be replaced by

$$[\rho s / (\rho v + \rho' v')], \quad (4.31)$$

where ρ', v' are the density and the sound velocity in the liquid. Since s at the core-mantle interface is larger than at the outside surface, while the area is smaller, the energies absorbed at the two surfaces are roughly equal. Depending on the frequency, the seismic signals from the two surfaces may interfere constructively or destructively at the position of any particular seismometer.

V. CONCLUSIONS

A realistic estimate⁵ of the background noise in a seismometer at a quiet location gives for the rms linear displacement

$$\langle y \rangle = 10^{-7} B^{1/2} \text{ cm}, \quad (5.1)$$

where B is the band width in hertz. This is valid for seismic background in the 1-Hz band, excluding intervals of reverberation after major earthquakes and storms. We are concerned with a displacement y_P obtained by correlating the seismic record over a few years with a sine wave at a particular pulsar period. It is not necessary that the intrinsic period of the pulsar remain constant, provided that any secular variations of the period are monitored. In any case, the apparent period varies annually because of the orbital motion of the Earth. In processing the data to obtain y_P , the sine wave must be frequency-modulated so as to keep in step with the actual pulsar signals observed at the same time. Provided that the relative timing of seismic and pulsar signals is maintained within a small fraction of a second, the effective band width of the signal y_P for a record extending over a few years is

$$B = 10^{-8} \text{ hertz}, \quad (5.2)$$

whether or not the pulsar period is constant to this accuracy. Hence the rms noise level for y_P is

$$\langle y \rangle_P = 10^{-11} \text{ cm}. \quad (5.3)$$

If we use an array of 100 seismometers and average the signals coherently, we obtain

$$\langle y \rangle_P = 10^{-12} \text{ cm}. \quad (5.4)$$

The use of an array may improve the noise level by substantially more than a factor of 10, since we are picking out vertically propagating waves and most of the background noise is horizontal or oblique.⁶ In any case, equation (5.4) is a safe estimate of the noise level obtainable with present-day equipment and data processing.

The comparison of equation (5.4) with equation (3.23) shows that the expected response to pulsars is about a factor of 10^5 too small to be detected. This estimate is based

⁵ See "A Discussion on Recent Advances in the Technique of Seismic Recording and Analysis," in *Proc. Roy. Soc.*, **A290**, 287–476 (1966), especially the section "Teleseismic Signal Extraction," by M. M. Backus, pp. 343–367.

⁶ See n. 5.

on three main assumptions: (1) a source equivalent to a star of zero bolometric magnitude, (2) the absence of reflection or resonance effects in the seismic response, and (3) a noise level in vertically propagating waves equal to the omnidirectional average noise. All three assumptions could turn out to be pessimistic. For example, a binary star with components of solar mass describing a circular orbit with period 2 sec, situated at a distance of 30 pc, would give a flux of gravitational energy about 10^8 times as intense as a zero-magnitude star. The amplitude y_P would then be 10^4 times as large as we estimated from assumption 1. The upper limit on possible source intensity is set, not by the emission mechanism, but by the requirement that the source be reasonably long-lived. Similarly, assumptions 2 and 3 could be pessimistic by factors of 10 or more. If we are lucky in all three places, we might make up the necessary factor of 10^5 in the signal-to-noise ratio.

Finally, we should remember the history of radio astronomy, which was greatly hampered in its early stages by theoretical estimates predicting that few detectable sources should exist. The predictions were wrong because the majority of sources were objects unknown to optical astronomers at that time. Whenever a new channel of observation of the Universe is opened, we should expect to see something unexpected. For this reason above all, the seismic detection of pulsars is not as hopeless an enterprise as the calculations here reported would make it appear.

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REFERENCES

- Allen, C. W. 1955, *Astrophysical Quantities* (1st ed.; London: Athlone Press), p. 174.
 Landau, L. D., and Lifshitz, E. M. 1962, *The Classical Theory of Fields*, trans. M. Hamermesh (Reading, Pa.: Addison-Wesley Publishing Co.), chap. xi.
 Love, A. E. H. 1927, *A Treatise on the Mathematical Theory of Elasticity* (4th ed.; Cambridge: Cambridge University Press), chaps. iii and xii.
 Maran, S. P., and Cameron, A. G. W. 1968, *Phys. Today*, **21**, No. 8, 41.
 Weber, J. 1961, "General Relativity and Gravitational Waves," *Interscience Tracts on Physics and Astronomy*, No. 10 (New York: Interscience Publishers).
 ———. 1966, *Phys. Rev. Letters*, **17**, 1228.
 ———. 1967, *ibid.*, **18**, 498.
 ———. 1968a, *ibid.*, **20**, 1307.
 ———. 1968b, *ibid.*, **21**, 395.
 Weber, J., and Larson, J. V. 1966, *J. Geophys. Res.*, **71**, 6005.
 Wentzel, G. 1949, *Quantum Theory of Fields*, trans. C. Houtermans and J. M. Jauch (New York: Interscience Publishers), chap. i.